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Anomaly free U(1) chiral gauge theories on a two dimensional torus

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Abstract

We consider anomaly free combinations of chiral fermions coupled to $U(1)$ gauge fields on a 2D torus first in the continuum and then on the lattice in the overlap formulation. Both in the continuum and on the lattice, when the background consists of sufficiently large constant gauge potentials the action induced by the fermions varies significantly under certain singular gauge transformations. "Ruling away" such discontinuities cannot be justified in the continuum framework and does not naturally fit on the lattice. Complete gauge invariance in the continuum can be restored in some models by choosing special boundary conditions for the fermions. Evidence is presented that gauge averaging the overlap phases in these models produces correct continuum results.

1. Introduction

Two dimensional chiral gauge theories provide a convenient testing ground for non-perturbative regularizations since they are much simpler to simulate numerically than four dimensional ones. As in four dimensions, there are anomalies in two dimensions that have to be cancelled and anomaly free theories can have global charges that are broken by topologically non-trivial gauge fields. This paper is about abelian chiral gauge theories with several Weyl fermions in Euclidean space–time. The left handed Weyl fermions, labeled by j, have charge q_{Lj} and the right handed ones, labeled by i, have charge q_{Ri}. Anomalies are cancelled by requiring $\sum_i q_{Ri}^2 = \sum_j q_{Lj}^2$. In order to avoid infra-red problems we consider these theories on a compact manifold; we prefer the gravitationally flat background of the torus. Invariance under rotations by 90 degrees is maintained by setting both sides of the torus equal to a common length, l.

Asymptotic freedom implies that perturbation theory works well when l is small relative to the model's intrinsic scales. On small tori the field strength is also small, so the gauge potentials are essentially constant up to gauge transformations. Further, the probability for nonzero topology is suppressed.

In the classical limit the gauge fields are connections on a bundle over the torus. For our purposes it is convenient to describe the torus utilizing four square patches of identical size. Each patch intersects every other patch in two separate intersections. The gauge fields defined locally on the different patches are gauge related on their intersections by transition functions.

The quantum gauge fields are close to a classical connection on a trivial bundle with zero curvature. The gauge invariant content of this configuration is expressed by two phases (angles), $e^{2\pi i h_\mu}, \mu = 1, 2$. One may associate these phases with two Polyakov loops wrapping once around the torus in the two directions. We consider two gauges: In the "uniform" gauge the transition functions are all unity and all connections are constant. We have $A_\mu = 2\pi h_\mu / l, \mu = 1, 2$ uniformly on all patches. In the "singular" gauge the connections are all zero and the phases reside entirely in the transition functions. The transition functions are nontrivial on two narrow strips winding around the torus in two directions and are chosen to produce the correct phase factor for parallel transport along any loop.

Consider one of the several Weyl fermions in any of the above backgrounds. The associated Grassmann path integration generates an effective action that depends on the gauge field. The perturbative anomaly vanishes since the curvature is zero. On a torus the set of continuous $U(1)$ gauge transformations splits into an infinite collection of disconnected

pieces labeled by the number of windings in the two directions. The absence of a perturbative anomaly does not ensure invariance under winding gauge transformations. Consider the winding gauge transformations which map a "uniform" gauge $A_\mu = 2\pi h_\mu / l$ to another "uniform" gauge A_μ with the h_μ shifted by some integers. Such a gauge transformation does not affect the "singular" gauge representation. Therefore, if the effective action is defined in the "singular" gauge, gauge invariance under these winding gauge transformations is automatic. However, if the effective action is defined in the "uniform" gauge, gauge invariance under the winding gauge transformations is not guaranteed. It turns out that if the Feynman diagrams are summed in the "uniform" gauge (using zeta-function regularization for example), the effective action indeed changes under a winding gauge transformation. (Any effective action that is smooth in the constant gauge potentials, that obeys some natural discrete symmetries and that reproduces all convergent Feynman diagrams will end up violating the winding symmetries.) When the contributions from all the Weyl fermions are summed up invariance under the winding gauge transformations is restored as a consequence of $\sum_i q_{Ri}^2 = \sum_j q_{Lj}^2$. The winding gauge transformations do not directly violate gauge invariance in either the "singular" or the "uniform" gauges. However, the imaginary part of the the effective action in the "singular" gauge differs from the one in the "uniform" gauge. The gauge transformation connecting a "singular" background to a "uniform" one is singular.

We concluded that even after perturbative anomalies have been cancelled some gauge non–invariance still afflicts the effective action. We used several patches to describe the gauge fields although there is no non-trivial topology. As physicists, we wish to rephrase the description employing no patches. Insisting on using functions defined over the whole torus, we end up describing the "singular" gauge configurations by potentials $A_\mu(x)$ that have linear δ-function singularities in the μ direction. As a result, the dimension of the gauge field part of the Dirac operator equals that of the derivative part and new effects can arise when the ultraviolet behavior is regulated. In a purely vectorial theory the problem disappears since one can preserve exact pairwise phase cancelation between left and right movers. The problem disappears even in the chiral case if at least one of the phase factors is exactly unity, or if both phase factors are in some limited region around unity (the actual size of the region is model dependent). In summary, the ingredients necessary to observe the gauge invariance violation are:

- A gauge background that has two nontrivial $e^{2\pi i h_\mu}$'s with values not too close to unity.
- A gauge transformation that has a singularity whose effect is to concentrate the parallel transporters around the torus to a line transverse to the loop. There is nothing "random" about these gauge transformations. They are very "ordered" and perfectly

reasonable even in a continuum setting.

In the continuum one may try to forbid "singular" gauge fields. This would be somewhat artificial, since it is not enforced by the action and since it would be unnecessary in the vector case. Such a decree makes little sense quantum mechanically, where the domain of integration in the functional integral is not restricted by ordinary smoothness properties. On a lattice, one lets the action govern the type of gauge configurations that appear and there is no natural way of imposing a restriction of this sort. The above problem will be encountered in any lattice regularization that avoids non-local gauge fixing. For example, if we used interpolating methods, following [1], gauge singularities of the linear δ-function type are abundant. If they are to be allowed the singularities we are focusing on certainly cannot be ruled out. Alternatively one may wish to restrict the phase factors, $e^{2\pi i h_\mu}$, to a range sufficiently close to unity. This does not seem consistent with translational invariance: In the computation of condensates in vectorial theories in topologically nontrivial backgrounds [2] the quantities lh_μ play the role of center of mass collective coordinates and one needs to integrate over the entire range of each h_μ in order to restore translational invariance. So, as long as we are on a torus, there seems to be no natural escape. However, on a sphere, or cylinder, it is possible that the problem can be avoided.

The above problem is a generic feature in the continuum and any reasonable regulator should reproduce the violation of gauge invariance. We shall show in some detail that the overlap formulation [3] has gauge violations in precisely the circumstances dictated by the continuum. A considerable effort has gone into convincing ourselves that these are, essentially, the *single* situations where consequential gauge breaking occur in the overlap. In particular, just the mere randomness of the gauge fluctuations creates no difficulties. On the contrary, according to Förster, Nielsen and Ninomiya [4] random gauge fluctuations will restore gauge invariance at long distances in any model with relatively mild gauge breaking in the ultraviolet.

Originally we planned to test the overlap non–perturbatively for abelian chiral models defined on the torus [3]. The present paper shows that most models do not provide a clean testing ground. Nevertheless, our results confirm the overlap as a valid regularization of chiral gauge theories. In particular, we construct a special class of abelian chiral models on the torus where gauge breaking are completely eliminated in the continuum. This is achieved by a judicious choice of fermionic boundary conditions. The lattice overlap reproduces the good continuum behavior correctly. If we replaced the boundary conditions in these models by simpler prescriptions a non–discriminating lattice test would indicate some persistent gauge breaking effects which could be misinterpreted as a lattice problem. We wish to stress again our view that these effects are not a lattice deficiency but a

continuum one.

The paper is organized as follows. In section 2, a specific chiral $U(1)$ model is defined in the continuum with anti-periodic boundary conditions for all fermions. In section 3, the chiral determinant is studied in the presence of constant gauge fields and the associated gauge transformations. Explicit examples of gauge field configurations show that the theory is not gauge invariant unless the zero modes of the gauge fields are sufficiently small. In section 4, the overlap formulation on the lattice of the theory in section 2 is described. In section 5, we show that the overlap reproduces the lack of gauge invariance found in section 3. In section 6 we discuss extensively gauge averaging along orbits. We assess the statistical significance of the violation of gauge invariance and investigate whether relatively large phase factors in conjunction with singular gauge transformations are its main source. Our objective is to show that employing gauge averaging along orbits as a way to restore exact gauge invariance on the lattice is a valid method. As long as there are no gauge violations in the continuum one recovers the desired outcome, and, reassuringly, if such gauge violations do exist, a clear signal is obtained. In section 7 we change the chiral model by picking new boundary conditions for the fermions. Now complete gauge invariance holds in the continuum and the overlap formalism in conjunction with gauge averaging produces correct results on all gauge orbits. In section 8 we summarize our findings and draw some conclusions. Readers who prefer a slightly more detailed overview now should skip to section 8 before reading sections 2 through 7.

2. A 11112 model

For definiteness we consider a specific $U(1)$ chiral model. All results can be carried over to any other $U(1)$ chiral model. We pick $q_{Li} = 1, i = 1, 2, 3, 4$ and a single right handed fermion with charge $q_R = 2$. Numerically, it is more economical to deal with this model than with the more traditional "345" model. The global chiral $SU(4)$ symmetry among the left-handed fermions in the "11112" model is of potential interest while the 345 model has no global nonabelian symmetry. Also, the fermion number violating 't Hooft vertex is slightly simpler in our case than in the 345 model [3].

The partition function of our 11112 model is given by

$$\mathcal{Z} = \int [dA_\mu][d\bar{\psi}][d\psi] \exp(-S[\bar{\psi}, \psi, A_\mu]) \tag{2.1}$$

$$S[\bar{\psi}, \psi, A_\mu] = S_g[A_\mu] + S_f[\bar{\psi}, \psi, A_\mu] \tag{2.2}$$

$$S_g[A_\mu] = \frac{1}{4e^2} \int d^2x F_{\mu\nu}^2 \tag{2.3}$$

$$S_f[\bar{\psi}, \psi, A_\mu] = -\sum_{k=1}^{4} \int d^2x \bar{\psi}_k^L \sigma_\mu (\partial_\mu + iA_\mu)\psi_k^L - \int d^2x \bar{\psi}^R \sigma_\mu^* (\partial_\mu + 2iA_\mu)\psi^R \tag{2.4}$$

$\sigma_1 = 1$ and $\sigma_2 = i$ in (2.4). The base manifold is an $l \times l$ torus and the fermion fields obey anti-periodic boundary conditions.

e and A_μ have dimensions of mass. The field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \epsilon_{\mu\nu} E$ has dimensions of mass squared and the topological invariant $Q_T \equiv \frac{1}{2\pi} \int d^2x E$ can take any integer value. The quantization of Q_T gives a precise meaning to the magnitudes of the charges assigned to the fermions. The path integral includes a sum over all topological sectors but, as outlined in the introduction, we will restrict ourselves to $Q_T = 0$.

In the zero topological sector we use functions to describe the gauge potentials. The gauge fields are decomposed as

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \phi - i g^*(x) \partial_\mu g(x) + \frac{2\pi}{l} h_\mu \tag{2.5}$$

h_1 and h_2 are zero modes of the gauge fields, i.e., $h_\mu = \frac{1}{2\pi l} \int d^2x A_\mu$. The electric field density is

$$E(x) = \partial^2 \phi(x). \tag{2.6}$$

Given $E(x)$, (2.6) can be used to solve for $\phi(x)$. $\phi(x)$ is a periodic function on the torus with a vanishing zero mode. $g(x)$ is a $U(1)$ valued function. h_1, h_2 and $E(x)$ constrained by $Q_T = 0$ constitute the gauge invariant content of the gauge field. The integral over A_μ can be replaced by an integral over $\phi(x)$, h_1, h_2 and $g(x)$. The first three label different gauge orbits and the last one labels points on a gauge orbit.

Only small and relatively slowly varying ϕ fields are allowed by S_g. However, S_g is independent of the h_μ's and has no control over their fluctuations. The probability distribution of the h_μ's is only influenced by the fermions through the induced effective action. In a finite Euclidean volume there is no mechanism to "freeze" these variables.* Therefore, once we establish that for some range of h_μ's we have a problem with gauge invariance, the issue cannot be discounted on the grounds that there is no weight for these backgrounds.

$g(x)$ itself is an arbitrary function, no condition on its magnitude or variability is imposed; in other words we include gauge singularities. As described in the introduction one can always split the torus into patches so that $g(x)$ is smooth in each patch. Discontinuities in $g(x)$ between patches can be stored in the transition functions between the

* For example, if one direction were infinite one of the h_μ's would have to minimize the effective potential and the problematic gauge backgrounds would disappear.

patches. The Dirac-Weyl operators act on sections and are smooth mappings of sections into other sections on the same coordinate bundle. We also can define a perfectly reasonable eigenvalue problem for the Weyl-Dirac operator although the eigenvalues are not scalars under rotations. In summary, allowing $g(x)$ to have discontinuities is both reasonable and mathematically acceptable. Of course, defining the chiral *determinants* requires care in general, and even more so when gauge singularities are present. But this is the task of renormalized quantum field theory.

3. Chiral determinant when $\phi = 0$

For $\phi = 0$, $S_g = 0$ and all values of h_μ along with all their possible gauge transforms are equally likely for the pure gauge action. In this section we study the chiral determinant when $\phi = 0$. We start with $A^u_\mu(x) = \frac{2\pi i}{l} h_\mu$. The superscript u means that A_μ is in the "uniform" gauge. The h_μ's in the range $[-1/2, 1/2)$ are gauge inequivalent. All other values of h_μ can be obtained by gauge transformations. Apart from the gauge transformations that take $h_\mu \to h_\mu + n_\mu$, we will also consider gauge transformations that result in the singular gauge field configuration, $A^s_\mu(x) = 2\pi i h_\mu \delta(x_\mu)$ (no sum on μ). Let $D^u(h_1, h_2)$ and $D^s(h_1, h_2)$ denote the chiral determinant of a single left handed Weyl fermion in the two backgrounds A^u_μ and A^s_μ.

The result for D^u is already known [5]. It can be obtained by zeta function regularization and other continuum methods.** Although the electric field vanishes, D^u is anomalous, not being invariant under $h_\mu \to h_\mu + n_\mu$. The explicit formula for D^u is

$$\begin{aligned}\frac{D^u(h_1,h_2)}{D^u(0,0)} &= \\ e^{i\pi h_2(h_1+ih_2)} &\frac{\prod_{n=1}^\infty [(1+e^{-2\pi(n-1/2)-2\pi i h_1+2\pi h_2})(1+e^{-2\pi(n-1/2)+2\pi i h_1-2\pi h_2})]}{\prod_{n=1}^\infty [(1+e^{-2\pi(n-1/2)})^2]}\end{aligned} \tag{3.1}$$

From (3.1) one obtains

$$D^u(h_1+n_1, h_2+n_2) = e^{i\pi(n_1 h_2 - n_2 h_1)} D^u(h_1,h_2) \tag{3.2}$$

The lack of gauge invariance arises because the determinant was made a smooth function of h_1 and h_2. Requiring smoothness for the uniform background is natural since the fermions

** In zeta-function regularization it is easiest to obtain an answer for the h_μ restricted to $[-1/2, 1/2)$ and then the answer is holomorphic in $h_1 + ih_2$. To restrict the gauge violation to the imaginary part of the effective action one needs to add a real quadratic term which breaks holomorphy.

see locally a field that depends linearly on h_1 and h_2. Also, in a finite Euclidean volume there is no thermodynamic mechanism to induce a singularity.

Now let us turn our attention to $A^s(x)$ and the associated D^s. Again one can compute the determinant by Feynman diagram methods. For h_1 and h_2 close to zero we expect the result to match D^u, but it should not change under $h_\mu \to h_\mu + n_\mu$ because the fermion sees only the transition functions $e^{2\pi i h_\mu}$. We conclude that in the fundamental domain

$$D^s(h_1, h_2) = D^u(h_1, h_2) \qquad \forall h_\mu \qquad -\frac{1}{2} \leq h_1, h_2 < \frac{1}{2} \tag{3.3}$$

and, for all integers n_μ,

$$D^s(h_1 + n_1, h_2 + n_2) = D^s(h_1, h_2). \tag{3.4}$$

The formulae for D^u and D^s do not yield a gauge invariant determinant for the 11112 model. In a uniform background field the partition function of all the fermions is

$$Z^u(h_1, h_2) = \Big[D^u(h_1, h_2)\Big]^4 \Big[D^u(2h_1, 2h_2)\Big]^* \tag{3.5}$$

From (3.2) is follows that

$$Z^u(h_1 + n_1, h_2 + n_2) = Z^u(h_1, h_2) \tag{3.6}$$

showing that the effective action restricted to A^u is gauge invariant. The disappearance of the violation in (3.2) is due to the anomaly cancelation condition $1^2+1^2+1^2+1^2 = 2^2 \equiv 4$. For the singular gauge field the partition function is

$$Z^s(h_1, h_2) = \Big[D^s(h_1, h_2)\Big]^4 \Big[D^s(2h_1, 2h_2)\Big]^* \tag{3.7}$$

From (3.4) it follows that (3.7) also is gauge invariant. Let us compare the partition function for the singular field with that for the uniform field. Using (3.3) we replace (3.7) by

$$Z^s(h_1, h_2) = \Big[D^u(f(h_1), f(h_2))\Big]^4 \Big[D^u(f(2h_1), f(2h_2))\Big]^*, \tag{3.8}$$

where $f(h)$ is the function that takes h into the range $[-1/2, 1/2)$ by an appropriate shift by an integer. Only if $f(2h_\mu) = 2f(h_\mu)$ for $\mu = 1, 2$ do we have

$$Z^s(h_1, h_2) = Z^u(h_1, h_2) \tag{3.9}$$

(3.9) will hold if the h_μ are each in the segment $[-1/4, 1/4]$. The result is gauge invariant if the h_μ are sufficiently small but not if they are large.

The conclusion we arrived at is somewhat unexpected. It holds for any anomaly free chiral $U(1)$ model on the torus as long as all fermions obey anti-periodic conditions. The region of h_μ where gauge invariance is preserved will be a function of the model but there will always exist ranges of h_μ where gauge invariance is lost. In the following sections we will show that this result holds rigorously in the overlap regularization. Of course, many other regularizations would also reproduce the same effect.

4. Overlap formalism

We embed an $L \times L$ lattice in the continuum $l \times l$ torus. The lattice spacing a is given by $aL = l$. The parallel transporters

$$U_\mu(n) = e^{i\int_0^1 A_\mu(x+ta\hat{\mu})dt} \tag{4.1}$$

on the links of the lattice are constructed out of ϕ, g and h_μ as follows. Replacing the continuum $g(x)$ we attach to each lattice site n a $U(1)$ group variable $g(n)$. to the plaquette with corners at n, $n+\hat{\mu}$, $n+\hat{\nu}$ and $n+\hat{\mu}+\hat{\nu}$ we associate an angle $\phi(n)$, a discretization of the continuum $\phi(x)$. The $\phi(n)$'s are restricted by the condition $\sum_n \phi(n) = 0$. Then

$$U_\mu(n) = g(n+\hat{\mu})g^*(n)e^{i\epsilon_{\mu\nu}[\phi(n)-\phi(n-\hat{\nu})]}e^{i\frac{2\pi}{L}h_\mu} \tag{4.2}$$

It is best to visualize the ϕ variables as living on the sites of the dual lattice. The Wilson gauge action is

$$\begin{aligned} S_g^w =& \frac{1}{e^2}\sum_n \mathrm{Re}[1 - U_2(n)U_1(n+\hat{2})U_2^*(n+\hat{1})U_1^*(n)] \\ =& \frac{1}{e^2}\sum_n [1-\cos((\partial_1^*\partial_1 + \partial_2^*\partial_2)\phi(n))] \end{aligned} \tag{4.3}$$

where ∂_μ are the forward lattice derivatives and ∂_μ^* are the backward lattice derivatives. The electric field per plaquette is $E(n) = (\partial_1^*\partial_1 + \partial_2^*\partial_2)\phi(n)$. The gauge invariant variable is the plaquette parallel transporter, $e^{iE(n)}$. The h_μ can be restricted to $[-1/2, 1/2)$ since the $g(n)$'s are arbitrary.

The fermionic path integral is defined on the lattice using the overlap formalism [3]. The continuum fermionic determinant for a left handed Weyl fermion with $q_L = 1$ is replaced by an inner product of two states:

$$\int d\bar{\psi}_L d\psi_L e^{-S_f(\bar{\psi}_L, \psi_L, A_\mu)} = {}_U^{\mathrm{WB}}\!< L - | L + >_U^{\mathrm{WB}} \tag{4.4}$$

The states $|L\pm>_U^{\rm WB}$ are the ground states of two many body Hamiltonians

$$\mathcal{H}^{\pm} = \sum_{n\alpha, m\beta} a^{\dagger}_{n,\alpha} \mathbf{H}^{\pm}(n\alpha, m\beta; U) a_{m,\beta}, \quad \{a^{\dagger}_{n,\alpha}, a_{m,\beta}\} = \delta_{\alpha,\beta}\delta_{nm} \tag{4.5}$$

$\alpha, \beta = 1, 2$ and $n = (n_1, n_2)$ with $n_\mu = 0, 1, 2, ..., L-1$. The single particle hermitian hamiltonians $\mathbf{H}^{\pm}$ are given by:

$$\begin{aligned}
\mathbf{H}^{\pm} &= \begin{pmatrix} \mathbf{B}^{\pm} & \mathbf{C} \\ \mathbf{C}^{\dagger} & -\mathbf{B}^{\pm} \end{pmatrix}, \\
\mathbf{C}(n, m) &= \frac{1}{2} \sum_{\mu} \sigma_\mu (\delta_{m,n+\mu} U_\mu(n) \ - \ \delta_{n,m+\mu} U^*_\mu(m)), \\
\mathbf{B}^{\pm}(n, m) &= \frac{1}{2} \sum_{\mu} (2\delta_{n,m} \ - \ \delta_{m,n+\mu} U_\mu(n) \ - \ \delta_{n,m+\mu} U^*_\mu(m)) \ \pm \ m\delta_{n,m}.
\end{aligned} \tag{4.6}$$

The parameter m is restricted only by $0 < m < 2$. The phases of the states $|L\pm>_U^{\rm WB}$ are chosen according to the Wigner-Brillouin convention, i.e. ${}_1< L\pm|L\pm>_U^{\rm WB}$ is real and positive. The determinant for a right handed Weyl fermion with $q_R = 1$ is

$${}_U^{\rm WB}< R-|R+>_U^{\rm WB} = \left[{}_U^{\rm WB}< L-|L+>_U^{\rm WB}\right]^*. \tag{4.7}$$

The $|R\pm>_U^{\rm WB}$ are the highest energy eigenstates of $\mathcal{H}^{\pm}$. The regulated partition function for the 11112 model on the lattice becomes

$$Z = \int [dU_\mu] e^{S_g^w} \left[{}_U^{\rm WB}< L-|L+>_U^{\rm WB}\right]^4 {}_{U^2}^{\rm WB}< R-|R+>_{U^2}^{\rm WB} \tag{4.8}$$

5. Overlap when $\phi = 0$

In this section we will show that the results of section 3 are reproduced in the overlap formalism. To this end we consider the lattice form of the uniform and singular gauge fields of section 3. The link variables corresponding to the uniform field are

$$U^u_\mu(n) = e^{\frac{2\pi i}{L} h_\mu} \tag{5.1}$$

while the ones replacing the singular field are

$$\begin{aligned}
U^s_1(n_1, n_2) &= \begin{cases} e^{2\pi i h_1} & \text{if } n_1 = 0 \\ 1 & \text{elsewhere} \end{cases} \\
U^s_2(n_1, n_2) &= \begin{cases} e^{2\pi i h_2} & \text{if } n_2 = 0 \\ 1 & \text{elsewhere} \end{cases}
\end{aligned} \tag{5.2}$$

$U_1^s(0,n_2)$ and $U_2^s(n_1,0)$ represent the transition functions in the continuum. We will denote the lattice overlap corresponding to a left-handed fermion by $D^u_{\rm lat}(h_1,h_2;L)$ and by $D^s_{\rm lat}(h_1,h_2;L)$ for the uniform and singular field respectively. For integers n_1 and n_2 it follows from (5.1) that

$$D^u_{\rm lat}(h_1,h_2;L) = D^u_{\rm lat}(h_1+n_1L,h_2+n_2L;L) \tag{5.3}$$

whereas from (5.2) one concludes that

$$D^s_{\rm lat}(h_1,h_2;L) = D^s_{\rm lat}(h_1+n_1,h_2+n_2;L). \tag{5.4}$$

$D^u_{\rm lat}(h_1,h_2;\infty)$ has no periodicity and reproduces the continuum result $D^u(h_1,h_2)$ in (3.1). In figure 5.1 we show how the limit $L\to\infty$ is approached. We have picked definite values for the h_μ's, namely $h_1=0.23$ and $h_2=0.37$. We then evaluated $D^u_{\rm lat}(0.23,0.37+n_2;L)$ as a function of n_2 for two values of L, $L=20$ and $L=60$. As one can see, except for $n_2=\pm L/2$, all points lie on the continuum line given by (3.2). The overlap for the uniform field approaches the continuum result for all values of h_μ.

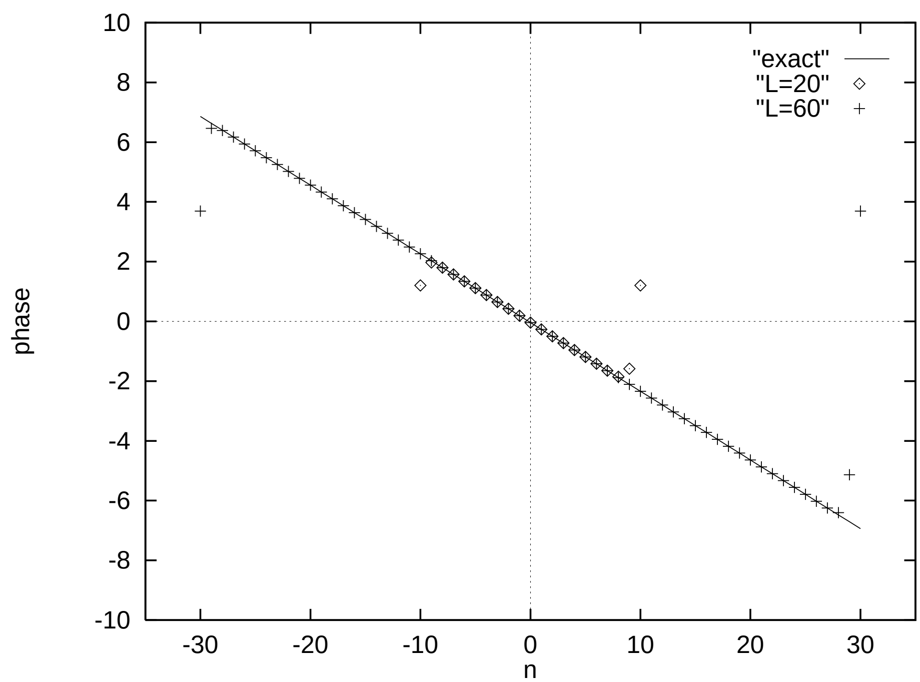

Figure 5.1 $D^u_{\rm lat}(0.23,0.37+n_2;L)$ as a function of n_2 for $L=20$ and $L=60$ along with the continuum result from section 3 (cf eqn(3.1)).

On the other hand $D^s_{\rm lat}(h_1,h_2;L)$ is periodic in h_μ with period 1 for all L. This is in agreement with (3.4). We illustrate that $D^u_{\rm lat}(h_1,h_2;\infty)$ and $D^s_{\rm lat}(h_1,h_2;\infty)$ reproduce

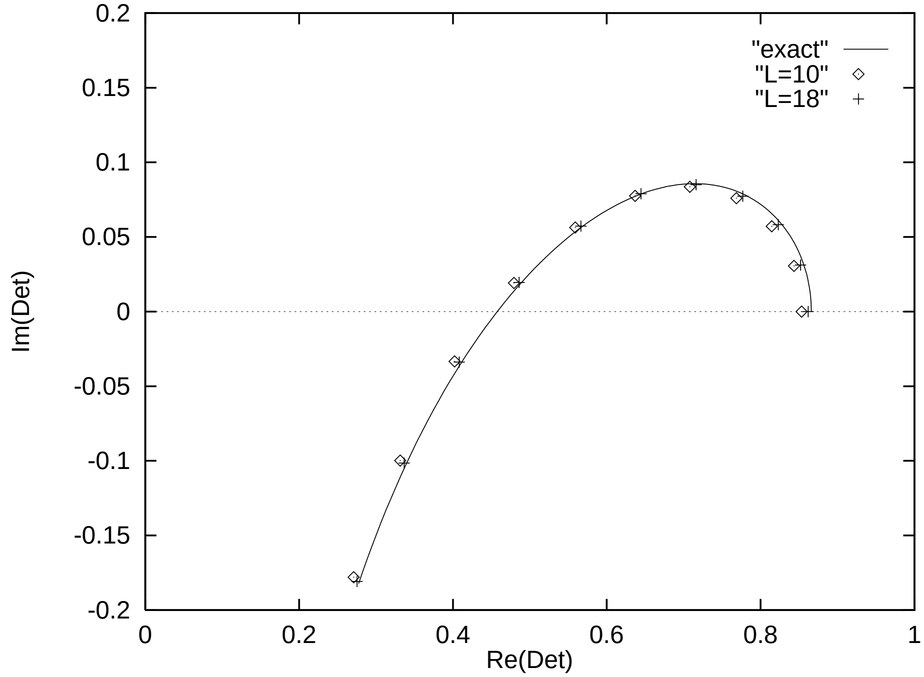

Figure 5.2 Locus of $D^u_{\rm lat}(0.37, h_2; L)$ as h_2 is varied from 0 to 0.5 for $L = 10, 18$ along with the continuum result from section 3(cf eqn(3.1)).

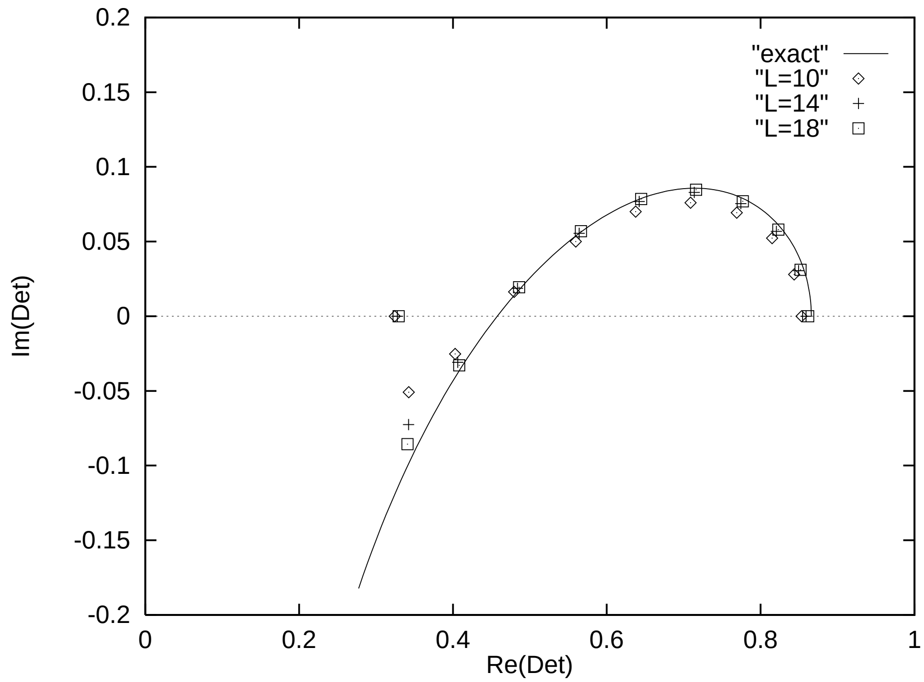

Figure 5.3 Locus of $D^s_{\rm lat}(0.37, h_2; L)$ as h_2 is varied from 0 to 0.5 for $L = 10, 14, 18$ along with the continuum result from section 3(cf eqn(3.1)).

their continuum counterparts by an example. We fix $h_1 = 0.37$ and evaluate for various

L's the lattice overlaps as a function of h_2 in the range $[0, 0.5]$. The overlap for $-h_2$ is related to h_2 by parity. In figure 5.2, we plot the locus of the complex valued function $D^u_{\rm lat}(0.37, h_2; L)$ as h_2 is varied from 0 to 0.5. We display $D^u(0.37, h_2, L))$ at 11 equally spaced values of h_2, for two values of L, $L = 10$ and $L = 18$. The continuum function $D^u(0.37, h_2)$ is also plotted. The lattice results approach the correct continuum limit.

Figure 5.3 is similar to figure 5.2, the singular $D^s_{\rm lat}(h_1, h_2; L)$ replacing the uniform $D^u_{\rm lat}(h_1, h_2; L)$ and displaying one more value of L, $L = 14$. Again, the known continuum limit is approached as L increases. The lattice effects are larger for the singular field than the uniform one. In particular the deviation from the continuum has to be large when h_2 is close to 0.5 because the continuum answer has a discontinuity there. On any finite lattice the discontinuity is approximated by a smooth function. As the ultraviolet cutoff increases this smooth function has an increasingly sharper profile around $h_2 = 0.5$.

6. Overlap along gauge orbits

The overlap formula for a single chiral fermion reproduces the ordinary anomaly [6] for continuum gauge fields and therefore is not gauge invariant. On the lattice there will be more terms, of higher dimension, that also break the gauge symmetry. For non-singular vector potentials the extra terms are local and vanish as the lattice spacing goes to zero at a fixed continuum external gauge field. When we combine various fermions like in the 11112 model to make up an anomaly free theory, the only gauge breaking left on the lattice comes from these extra terms. Such extra gauge breaking terms will inevitably appear in any approach where the fermion path integral factorizes into a product of factors, one for each irreducible multiplet. This factorization is a formal property of the continuum fermion path integral. In our previous work [3] we suggested dealing with the extra gauge breaking terms by simply averaging over each gauge orbit. If they are not too large, and if anomaly free chiral gauge theories in the continuum exist also beyond perturbation theory, the most plausible outcome is that the averaging along the orbit simply adds some irrelevant local gauge invariant terms to the rest of the action. For example, a gauge breaking term in an action for a pure gauge theory that has the form of a mass term for the gauge bosons, when averaged over the gauge orbits, induces only effects irrelevant in the infrared, as long as its coefficient is not too large [4].

In this section we shall repeatedly start from some configuration that has a typical gauge invariant content and average over its gauge orbit by computing the overlap for many gauge transformations of the original configuration. The overlap enjoys the nice property that all the gauge breaking is restricted to its phase [3]. Therefore, the gauge invariant modulus of the overlap can be pulled out from the integration along the gauge

orbit and all we have to do is to average the phases. The result of this averaging will be some complex number, $\mathcal{Z}$. $|\mathcal{Z}|$ might be of some interest academically, but it can carry only irrelevant information for the continuum target theory because the modulus of the overlap already contains all the needed real part of the effective action. We conclude that in practice one should discard $\log|\mathcal{Z}|$, the contribution of gauge averaging to the real part of the total fermion induced effective action. We only care about Φ, where $\mathcal{Z} = |\mathcal{Z}|e^{i\Phi}$. Φ depends only on the gauge invariant content of the initial configuration. Again a special property of the overlap comes in here: Just like in the continuum, Φ will be parity odd. If after subtracting from Φ the continuum value determined by the background we obtain a remainder that admits an expansion in local operators, we have an exact symmetry on the lattice that restricts these operators to being parity odd. (Of course, such a local expansion is an open possibility only if the perturbative anomalies cancel; otherwise, gauge averaging the exponent of the anomaly is bound to induce non-local terms.) There exist no operators of dimension two or less that are parity odd, local, and gauge invariant functionals of the background. Assume that the continuum theory is completely well defined and that we start the gauge averaging from a relatively smooth background. The dimensionalities of the allowed operators imply that Φ will converge to the continuum gauge invariant parity odd answer as we refine the lattice.

The simplest background one could imagine is certainly one in which there is no curvature and all Polyakov loops are trivial. The continuum phase Φ is supposed to vanish in this background. The lattice overlap on this "trivial orbit" ($\phi = 0$ and $h_\mu = 0$) for a single chiral fermion was proven to be real in [3].* Thus, on the trivial orbit everything

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* In particular this nullifies the concerns expressed by Golterman and Shamir in their publications dealing with the overlap. Since the literature has become somewhat entangled, we set things straight in this footnote to avoid confusing those readers who are aware of most recent publications in the field, but are not directly active in it. The first paper by Golterman and Shamir, GSI, [Phys. Lett. B353 (1995) 84] criticizing the overlap has two major errors. Both errors were anticipated in [3]. Since GSI incorrectly claimed a rigorous mathematical equivalence between the overlap and a specific waveguide model, we wrote a note [7] repeating section 6 in [3] which disproves this claim. We focused on only one of the errors in GSI since it related to the central result of GSI and showed that the rigorous equivalence claimed in GSI was false. After our note [7] was circulated, Golterman and Shamir published an erratum, GSII [Phys. Lett. B359 (1995) 422], in which they announced this error in GSI but claimed that GSI still had shown that the overlap would fail. Specifically, their claim was that, if one took a $U(1)$ chiral model in 2D

works fine (there could be some sign switches, but they are very rare). However, this might cease being the case when we move away from the trivial orbit because there the exact reality [3] no longer holds. We expect the fluctuations of the overlap along the orbit to gradually increase as the background contains more local electric field and less trivial Polyakov loop transporters. We would like these fluctuations to be insignificant in the continuum limit. We shall work at a finite physical size in Euclidean space-time. On the lattice the continuum limit is approached keeping eL fixed in (4.3) and taking L to infinity. The physical size of the torus is proportional to eL. As discussed in the introduction, we focus on small systems.

In this section we evaluate the overlap for the 11112 model in various situations in order to study Φ. We will show that if the h_μ are sufficiently small the integration over gauge orbits yields the physical result. If the h_μ are large, gauge violations of the type discussed in sections 3 and 5 wash out the correct value of Φ unless we exclude the "singular" gauge configurations.

Let us first consider the case when $h_\mu = 0$. eL controls the size of the fluctuations in ϕ. If eL is small, ϕ is close to zero and we expect the phase fluctuations along the orbit to be small. These fluctuations are expected to also decrease as we take L to infinity, approaching continuum at fixed eL.

These trends are illustrated in figures 6.1–6.3. The overlaps for randomly drawn sets $\{g(n)|n_1, n_2 = 1, ..., L\}_i$ only differ by a phase. Choosing the $g(n)$ randomly in $U(1)$, we computed this phase and subtracted from it the phase when all $g(n) = 1$. When all $g(n) = 1$ (Landau gauge) the phase is close to zero. Φ vanishes in the continuum. We denote the remainder phase for each set by Φ_i. The distribution of the Φ_i obtained for $i = 1, 1000$ is then plotted. The horizontal axis in all these plots is the value of the phase, Ψ, in units of π. The vertical axis is the probability of getting a particular Ψ, $p(\Psi)$. We compute the phase for each chirality individually. The left handed contribution is

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with four identical copies of its fermionic content and restricted the gauge background to the trivial orbit, one would discover that the gauge degrees of freedom labeling the points along the orbit will not decouple. This statement was repeated in GSIII [hep-th/9509027] and in other papers quoting GSI, GSII, GSIII. The statement is still wrong and this goes back to the second major error in GSI and GSII which was anticipated in section 12 of [3]. There it was proven that the overlap on the trivial orbit is real for any chiral fermion. In particular, for the model considered in GSI, one has absolute independence of the points on the orbit. This was pointed out in private to the authors before they wrote their erratum GSII.

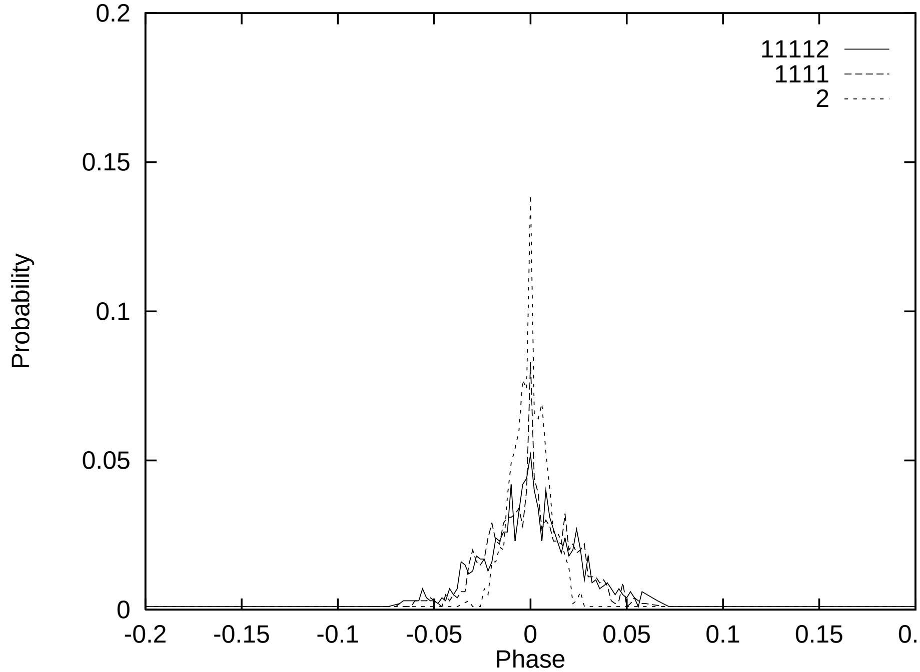

Figure 6.1 Normalized phase distribution for a sample gauge field at $eL = 0.1\pi$ with $h_1 = h_2 = 0$ on a 6×6 lattice.

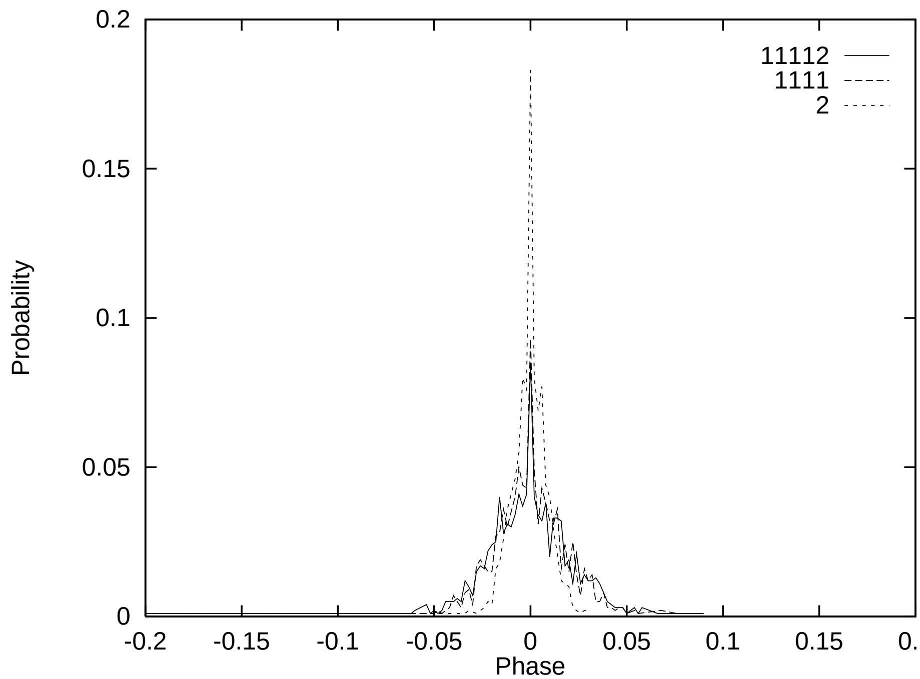

Figure 6.2 Normalized phase distribution for a sample gauge field at $eL = 0.1\pi$ with $h_1 = h_2 = 0$ on a 8×8 lattice.

contained in the points labeled by 1111 and the right handed contribution in the points

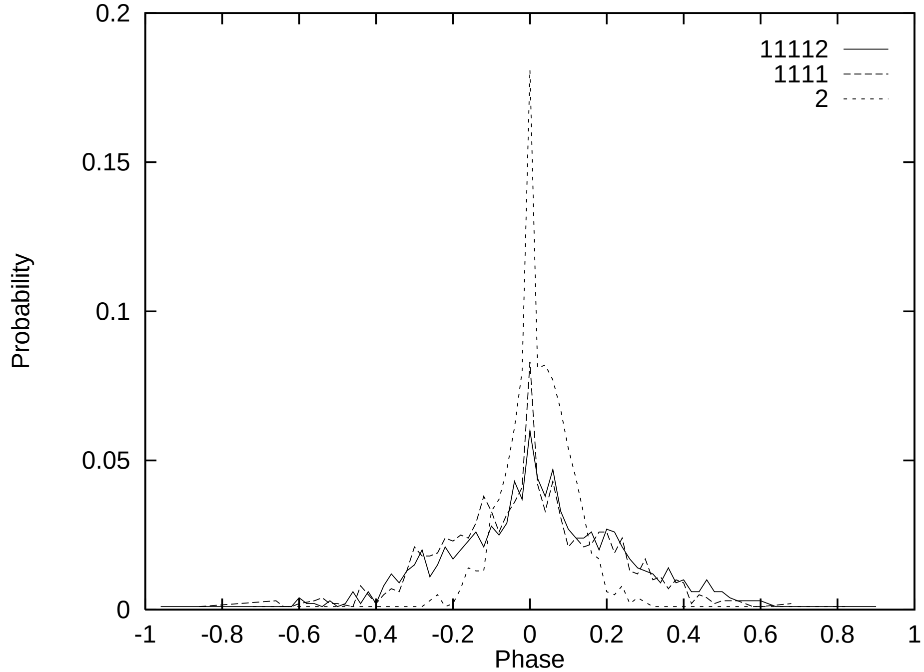

Figure 6.3 Normalized phase distribution for a sample gauge field at $eL = \pi$ with $h_1 = h_2 = 0$ on a 6×6 lattice.

labeled by 2. The phase of the product of the overlaps for the anomaly free combination is labeled by 11112. Only the 11112 points are of direct relevance, but the 1111 and 2 points indicate what an anomalous theory would yield, and show how anomaly cancelation works. In figure 6.1 we have fixed $eL = 0.1\pi$ and $L = 6$. In figure 6.2 we again have $eL = 0.1\pi$ but $L = 8$. Each time, a ϕ was drawn using the distribution in (4.3) which depends on $e = eL/L$. The fluctuations in 6.2 have decreased somewhat relatively to 6.1. In figure 6.3 we have increased eL to π and set $L = 6$. Now the fluctuations are larger than in figure 6.1. However, the fluctuations in these three figures are all small and all three distributions in each figure are well peaked around 0. Thus, figures 6.1–6.3 do not exhibit any sizable numerical difference between anomalous and anomaly free theories. This indicates that the specific lattice gauge breaking effects, although not large in an absolute sense, are larger numerically than the gauge breaking effects induced by the continuum anomaly, fermion by fermion.

To attain an indication of what kind of gauge transformations are responsible for the larger effects we sample the points on the orbit in a more controlled manner. We introduce a parameterization $g(x) = e^{i\kappa\chi(x)}$ and draw the $\chi(x)$ from a distribution similar to (4.3):

$$S_{\mathrm{gf}} = \frac{(eL)^2}{(e\pi)^2} \sum_n [1 - \cos((\partial_1^* \partial_1 + \partial_2^* \partial_2)\chi(n))] \tag{6.1}$$

This amounts to adding a $(\partial_\mu A_\mu)^2$ gauge fixing term in the continuum. Large momentum modes are suppressed somewhat and κ controls the amplitude. As κ is increased one spans the whole gauge orbit. The added term to the action corresponds to a "soft" gauge fixing term. In principle we do not want any "soft" or δ–function gauge fixing at this stage when the regularization method itself is tested. However, when a simulation with the objective of obtaining numerical results is carried out keeping such gauge fixing terms might be useful. Here we added this term just because it facilitates a better understanding of the difference between anomalous and anomaly free theories. In figures 6.4 and 6.5 we plot the phase distribution for two values of κ at $eL = \pi$ and $L = 6$. In figure 6.4, $\kappa = 0.1$ and in figure 6.5, $\kappa = 0.5$. The distribution for the 11112 model is much sharper than the ones for 1111 or 2 illustrating the anomaly cancelation. In conclusion, the fluctuations induced by the high momentum modes of the gauge transformations are larger than the ones induced by the their low momentum modes.

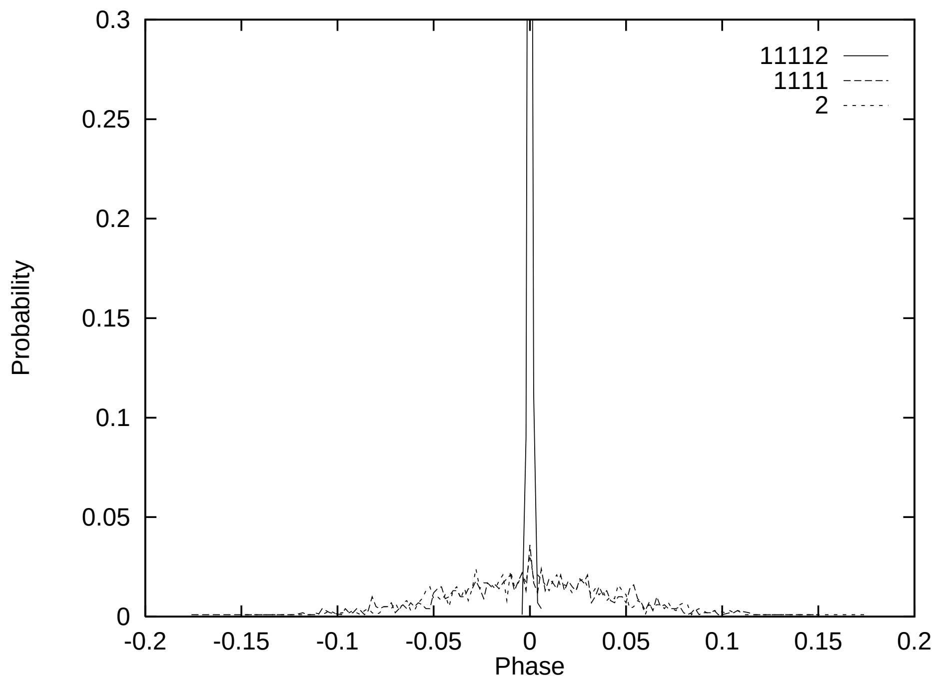

Figure 6.4 Normalized phase distribution for a sample gauge field at $eL = \pi$ and $\kappa = 0.1$ with $h_1 = h_2 = 0$ on a 6×6 lattice.

Now we proceed to study the phase distribution when $h_\mu \neq 0$ to tie in with the lack of gauge invariance observed in sections 3 and 5. If both the h_μ's are sufficiently small we do not expect large fluctuations. This is illustrated in figures 6.6 and 6.7. In both these figures we have set $\phi = 0$ and the points in the orbit are drawn randomly. In figure 6.6 we have set $h_1 = 0.07$ and $h_2 = 0.13$ whereas in figure 6.7 they are set at $h_1 = 0.23$

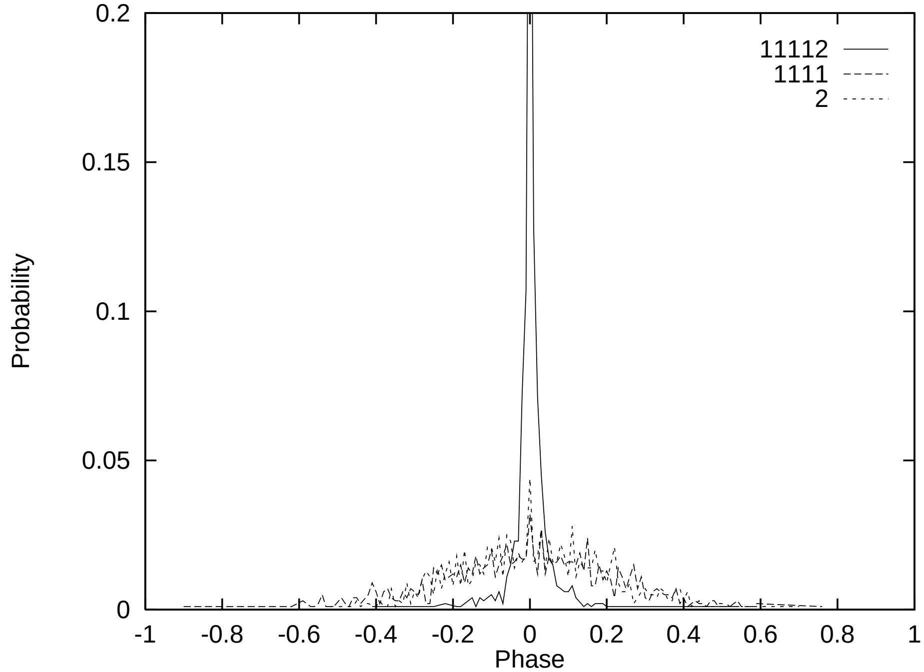

Figure 6.5 Normalized phase distribution for a sample gauge field at $eL = \pi$ and $\kappa = 0.5$ with $h_1 = h_2 = 0$ on a 6×6 lattice.

and $h_2 = 0.37$. The dramatic difference between the two cases is obvious. No qualitative change takes place if one also introduces a ϕ and this is shown in figure 6.8 where we have generated a non-zero ϕ by setting $eL = \pi$ and kept the h_μ's at the same values as in figure 6.7.

Figures 6.7 and 6.8 show that the distribution of phase is quite biased and uneven if the phase factors are large. In section 3 and 5 we gave a specific example whereby gauge invariance is violated in this case. Now we would like to address the issue as to what class of gauge transformations are responsible for the characteristics of the observed distribution. In particular we would like to know if the bias is all due to "singular" configurations of the type discussed in sections 3 and 5.

A first attempt is to introduce again a suppression of the high momentum components of the gauge transformations as we did with equation (6.1). This indeed removes the bias and unevenness, but is not a detailed enough test because we want to see that not all the randomness in the gauge degrees of freedom has to be reduced in order to eliminate the gauge violations. Randomness is needed for decoupling via the Förster–Nielsen–Ninomiya [4] mechanism. Only the very ordered large scale singularities are causing the gauge violations, and they are a feature of the continuum theory, not the lattice.

To achieve this sharper distinction we need a slightly different view of the basic idea

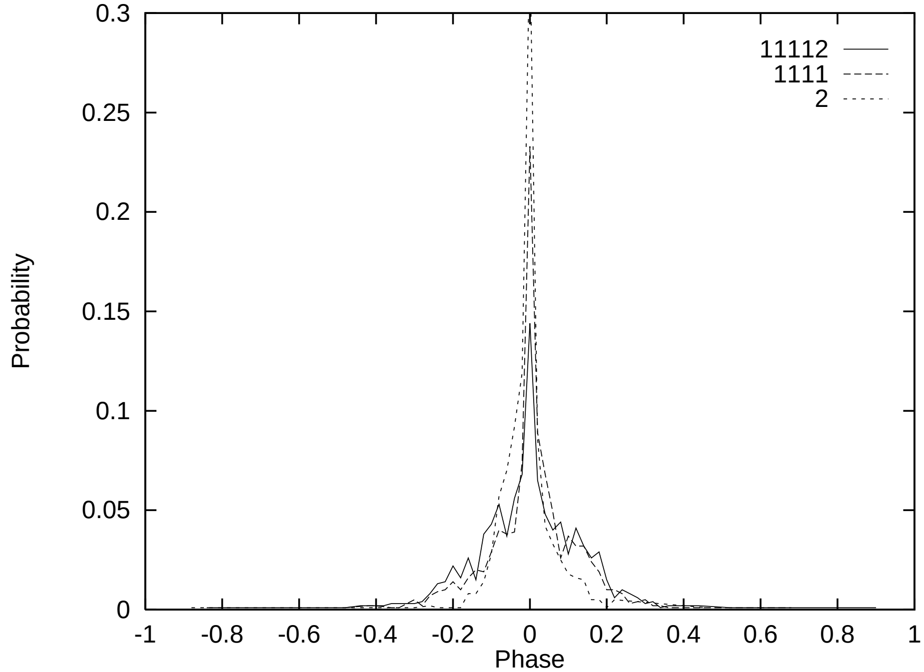

Figure 6.6 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.07$, $h_2 = 0.13$ on a 6×6 lattice .

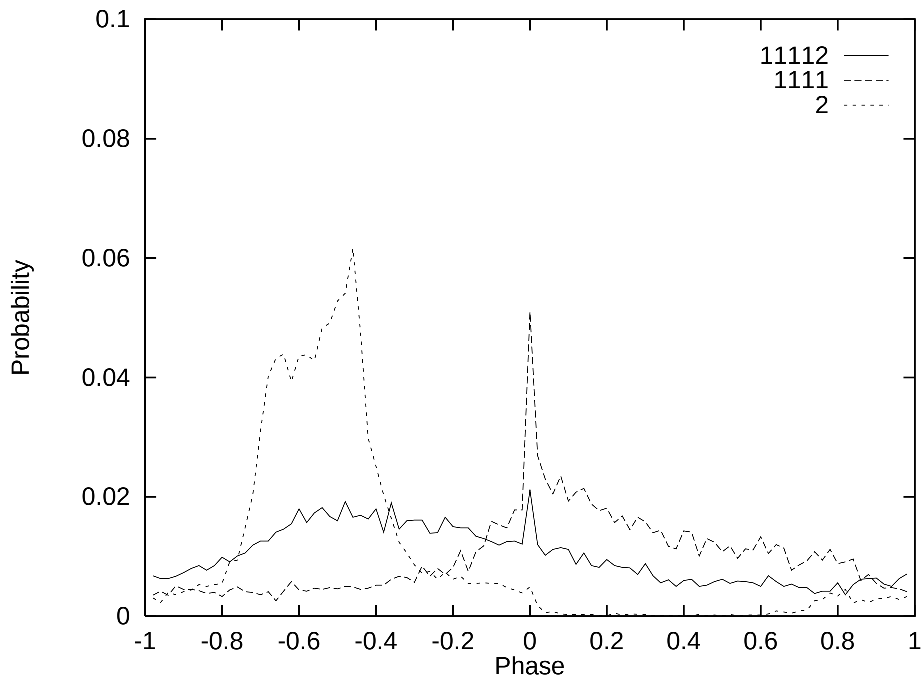

Figure 6.7 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.23$, $h_2 = 0.37$ on a 6×6 lattice .

of lattice regularization. The traditional point of view is that the U_μ variables correspond

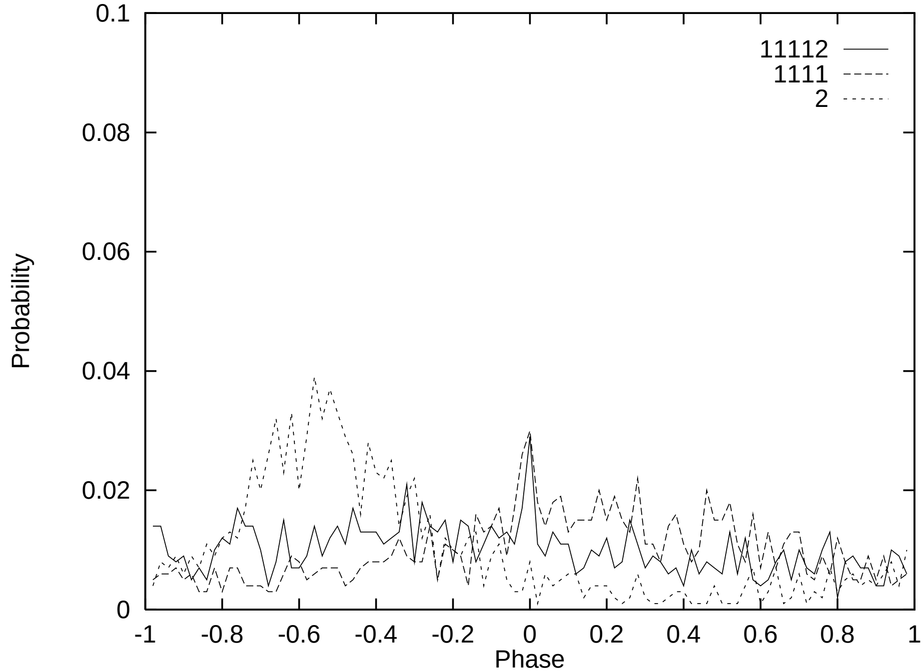

Figure 6.8 Normalized phase distribution for a sample gauge field at $eL = \pi$ with $h_1 = 0.23$, $h_2 = 0.37$ on a 6×6 lattice.

to path ordered integrals of some vector potential defined globally over the lattice. There exists another interpretation, due to Phillips [8]. The elementary cells of the dual lattice are viewed as a set of patches and the U_μ variables only represent the transition functions. The connections inside the patches are thought of as being all trivial. Which picture one adopts has no practical consequence in any of the present day applications of lattice gauge theory. In the context of chiral gauge theories however, the interpretation matters conceptually. It makes sense, in particular in view of our earlier discussion of the singular gauges, that a "patchy" view be maintained to some degree. It is not necessary to have one distinct patch per elementary cell, nor do the link variables have to all correspond to transition functions, or all correspond to connections. It does make sense that the sizes of the patches go to zero in the continuum so that arbitrary gauge singularities are included. Thus we could imagine that we have a certain number of lattice cells per patch (this number could even be taken to diverge in the continuum, but not as fast as the physical scales). Inside each patch one could impose a smoothness condition on the gauge transformations, or, even fix the gauge completely. This would be an entirely local operation, different in essence from the usual overall gauge fixing. The gauge transformations in distinct patches would still be totally unrestricted. In short, if the number of lattice sites is $\mathcal{N}$ and the gauge group is $\mathcal{G}$, the invariance group will no longer be $\mathcal{G}^{\mathcal{N}}$, but $\mathcal{G}^{\mathcal{N}'}$ with $\mathcal{N}/\mathcal{N}'$ being some finite

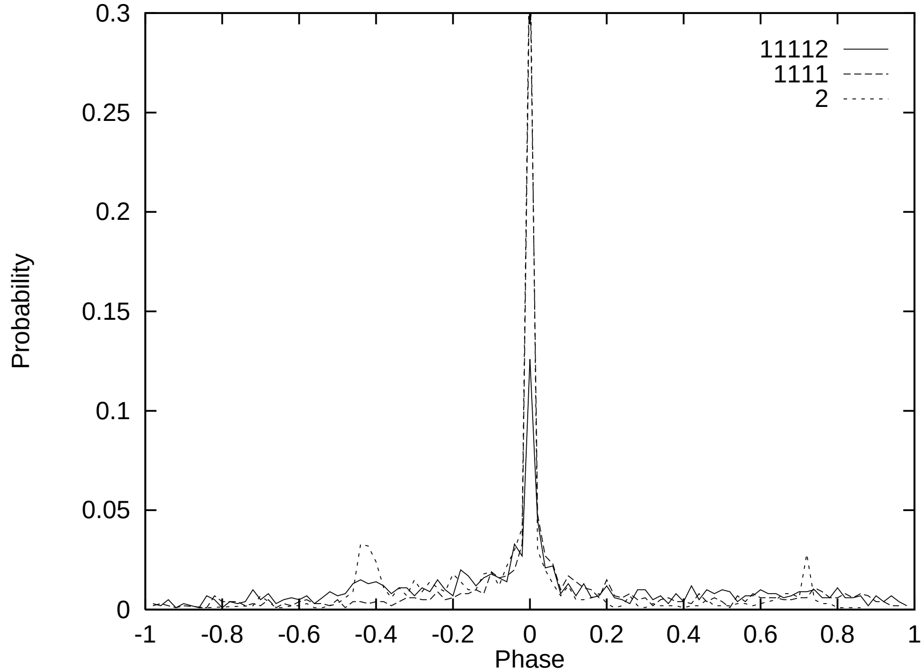

Figure 6.9 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.23$, $h_2 = 0.37$ when the points on the gauge orbit are restricted to be constant on blocks ($L = 9$, $L_b = 3$) and the starting point is the "uniform" configuration.

integer (or even diverging in the continuum). Such a setup would allow one to reduce the magnitude of the extra lattice gauge breaking terms with no need to construct continuum interpolations. Since the real part of the lattice version of the chiral determinant can be made exactly gauge invariant it is only the gauge transformations that might need some control.

The above general point of view can be easily realized in our application. We take an $L \times L$ lattice and divide it into non-intersecting $L_b \times L_b$ blocks. We think of each block as representing a two dimensional contractible patch. The links connecting the blocks represent transition functions and the links inside the block represent a gauge field in the local coordinates of that patch. Now we set ϕ to zero and consider two gauge realizations of the h_μ's (like in section 5) as initial configurations. As before, in (5.1-2), we refer to the first one as the "uniform" configuration and the second one as the "singular" configuration. Starting from the initial configuration we make gauge transformations with $g(n)$ identical on every site in a single block but taking random and independent values on distinct blocks. We picked $h_1 = 0.23$ and $h_2 = 0.37$. The chiral determinant for the 11112 model in this background is complex when we are in the "uniform" gauge, i.e. $\Phi \neq 0$.

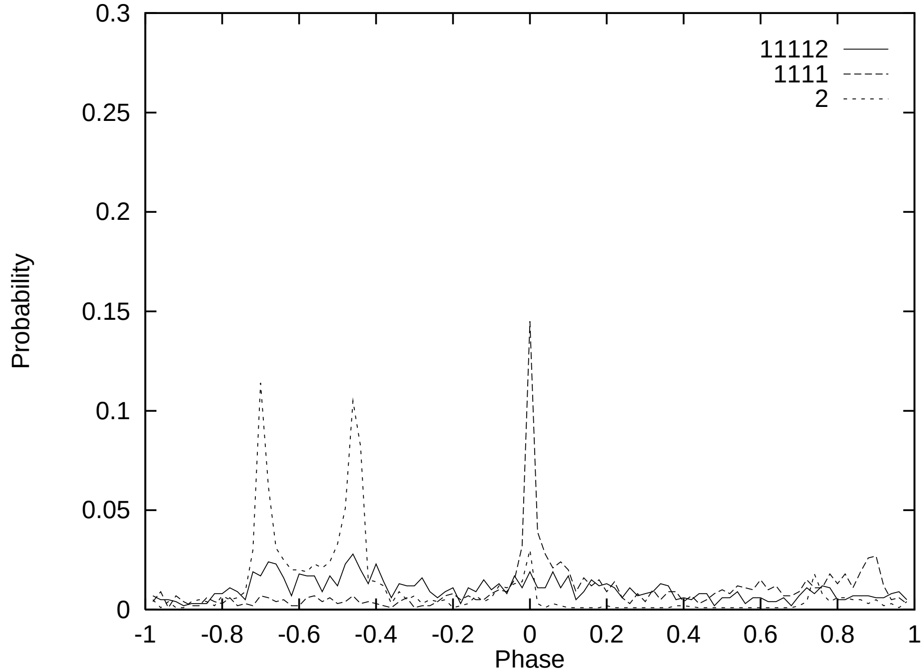

Figure 6.10 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.23$, $h_2 = 0.37$ when the points on the gauge orbit are restricted to be constant on blocks ($L = 9$, $L_b = 3$) and the starting point is the "singular" configuration.

We computed the distribution of the difference phases Φ_i when drawing random gauge transformations constrained per block as described above. In figure 6.9 we plot the results when the starting point was a "uniform" configuration and in figure 6.10 we plot the distribution when the starting point was a "singular" configuration. In both figures $L = 9$ and $L_b = 3$. Clearly the fluctuations in the "singular" case are qualitatively different. The gauge transformations we included did allow many singularities "between" the blocks. However, the restrictions did not allow the gauge transformations to "pile up" all of the Polyakov loop transporters on a single link (or very few links) in the uniform case. As a result figure (6.9) shows no significant gauge violations. However, in the singular case the restricted gauge transformations are allowed to "spread out" the Polyakov loops over many different block boundaries, (the location of the singularities in the initial configuration was picked at an ordered set of block boundaries compatible with (5.2)). Thus figure (6.10) does exhibit a significant amount of gauge violation. We conclude that there is evidence on the lattice that the real source for all important gauge violations is the one identified in the continuum and that there is no reason to suspect the lattice overlap of introducing new undesirable effects.

We now wish to show that gauge averaging along the orbits would perform satisfactorily for any background if we could forbid the singular gauge transformations. Since the electric field never caused any difficulties we consider only the case of large h_μ's but with $\phi = 0$. We restrict ourselves to a uniform initial configuration and do blocked gauge transformations as above. The phase fluctuations are controlled but larger than in the other cases of interest, namely $h_\mu = 0$ or small h_μ with non-zero ϕ. We fixed $L_b = 3$ and considered $L = 6, 9, 12, 15, 18, 21$. In each of these lattices we considered $10,000$ points on the orbit of blocked gauge transformations and averaged as before. The results for $\mathrm{Im}(\log < e^{i\pi\Phi_i} >)/\pi$ are tabulated in Table 6.1. The result for the 1111 or the 2 case appear to grow as L is increased whereas the result for the 11112 case is stable at some small value. Increasing the block size further decreases this small value. We see that indeed anomaly cancelation does play a role, as the right and left sectors by themselves do not seem to allow the lattice Φ to approach any limit, and definitely not the correct one. This indicates a restoration of gauge invariance in the 11112 model as we go to the continuum limit. The gauge transformations we included are very random on scales larger than the block size but they cannot bring the Polyakov loops to the "singular" gauge. While this proves our point we would not go as far as suggesting this as a possible redefinition of the 11112 model, simply because the decision about which initial configuration to pick involves a nonlocal choice.

Table 6.1

L	1111	2	11112
6	0.032 ± 0.007	-0.134 ± 0.008	-0.109 ± 0.014
9	0.147 ± 0.007	-0.207 ± 0.008	-0.089 ± 0.019
12	0.171 ± 0.013	-0.250 ± 0.004	-0.093 ± 0.019
15	0.215 ± 0.014	-0.289 ± 0.011	-0.083 ± 0.031
18	0.213 ± 0.023	-0.310 ± 0.012	-0.098 ± 0.044
21	0.238 ± 0.016	-0.308 ± 0.009	-0.083 ± 0.042

7. A new 11112 model

Up to now we restricted our attention to the 11112 model on the torus with antiperiodic boundary conditions for all fermions. This model has an $SU(4)$ flavor symmetry. We identified gauge orbits on which gauge invariance under certain singular gauge transformations cannot be maintained.

In this section we show that by allowing the fermionic boundary conditions to break the flavor $SU(4)$ we can arrange for both $Z^u(h_1, h_2)$ and $Z^s(h_1, h_2)$ (see (3.5) and (3.7))

to be periodic under $h_\mu \to h_\mu + \frac{1}{2}n_\mu$ with integer n_μ. One can then restrict the h_μ to the interval $[-\frac{1}{4}, \frac{1}{4}]$ ensuring $Z^u(h_1, h_2) = Z^s(h_1, h_2)$ for all h_μ (see (3.9)).

The new 11112 model easily generalizes to models containing one left handed fermion of charge Q and Q^2 right handed fermions of unit charge. We define the new models for arbitrary integer $Q > 0$. The left handed fermion, ψ_L obeys anti-periodic boundary conditions as before. It is convenient to use a doubled index to label the different flavors of the righthanded fermions: $\psi_{R,\alpha\beta}$ where $\alpha, \beta = 1, 2, ..., Q$. The boundary conditions for the ψ_R are

$$\begin{aligned} \psi_{R,\alpha\beta}(x_1 + lN_1, x_2 + lN_2) &= e^{\frac{2\pi i}{Q}(N_1\alpha + N_2\beta - \frac{N_1+N_2}{2})}\psi_{R,\alpha\beta}(x_1, x_2) \\ &= (-1)^{N_1+N_2} e^{\frac{2\pi i}{Q}[N_1(\alpha - \frac{1+Q}{2}) + N_2(\beta - \frac{1+Q}{2})]}\psi_{R,\alpha\beta}(x_1, x_2) \end{aligned} \tag{7.1}$$

In the old model the generalization of (3.5) to the charge Q case would have been:

$$Z^u(h_1, h_2) = [D^u(h_1, h_2)]^Q [D^u(Qh_1, Qh_2)]^* \tag{7.2}$$

In the new model the above is replaced by

$$Z^u_{\rm new}(h_1, h_2) = \prod_{\alpha,\beta=1}^{Q} \left[D^u(h_1 + \frac{\alpha}{Q} - \frac{1+Q}{2Q} \ , \ h_2 + \frac{\beta}{Q} - \frac{1+Q}{2Q}) \right] \ [D^u(Qh_1, Qh_2)]^* \tag{7.3}$$

Using the product representations of (3.1) it is not hard to prove that

$$\prod_{\alpha,\beta=1}^{Q} \left[D^u(h_1 + \frac{\alpha}{Q} - \frac{1+Q}{2Q} \ , \ h_2 + \frac{\beta}{Q} - \frac{1+Q}{2Q}) \right] = D^u(Qh_1, Qh_2) \tag{7.4}$$

Hence

$$Z^u_{\rm new}(h_1, h_2) = |D^u(Qh_1, Qh_2)|^2 \tag{7.5}$$

It is now obvious that $Z^u_{\rm new}(h_1, h_2) = Z^s_{\rm new}(h_1, h_2)$ since one has periodicity under independent shifts of each h_μ by integral multiples of $1/Q$. Moreover, all the imaginary parts of the continuum action vanish just like in a vectorial model. Nevertheless, the model is chiral as evidenced by the different charges carried by the left and right handed fermions. The model is definitely not vectorial even "in disguise" when Q is even since then the total number of Weyl fermions is odd.

On the lattice, the relevant product of overlaps will contain an imaginary part since the continuum cancelation between the right and left handed fermions no longer holds exactly. The unwanted, gauge breaking terms discussed in the previous section are present here also since the theory is not vectorial. If the overlap indeed produces correct results in

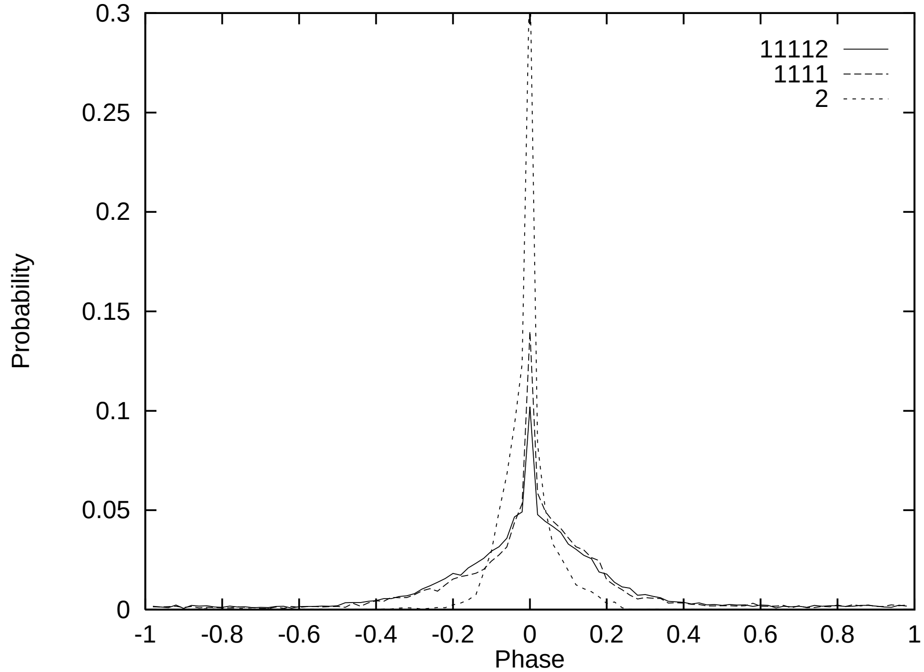

Figure 7.1 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.07$, $h_2 = 0.13$ on a 6×6 lattice.

the continuum limit after gauge averaging along the orbits we should see no longer any significant difference between backgrounds with relatively small values of h_μ and backgrounds with larger values. In particular, the qualitative difference between figures 6.6 and 6.7 in the previous section should disappear.

To check this we set $Q = 2$ as before. We then carried out numerical simulations similar to the ones that produced figures 6.6 and 6.7 but with the new model. The results are in figures 7.1 and 7.2, where 7.1 is the analogue of 6.6 and 7.2 replaces 6.7. While 6.6 and 7.1 are similar, 6.7 and 7.2 are quite different. In 7.2 we see clear evidence that the phases coming from the left handed and from the right handed fermion cancel during gauge averaging leaving us with an answer that is real (up to small ultraviolet and statistical effects).*

Therefore, the new chiral 11112 model is gauge invariant in the continuum under all gauge transformations on all gauge orbits and gauge averaging of the WB phases of the lattice overlap reproduces the expected results. We checked that turning on electrical fields again has no significant effect on the behavior of gauge averaging. Armed with this

* We also looked at the contribution to the real part of the effective action – which we discard – and note that in the new model the induced real part is numerically smaller then in the old model.

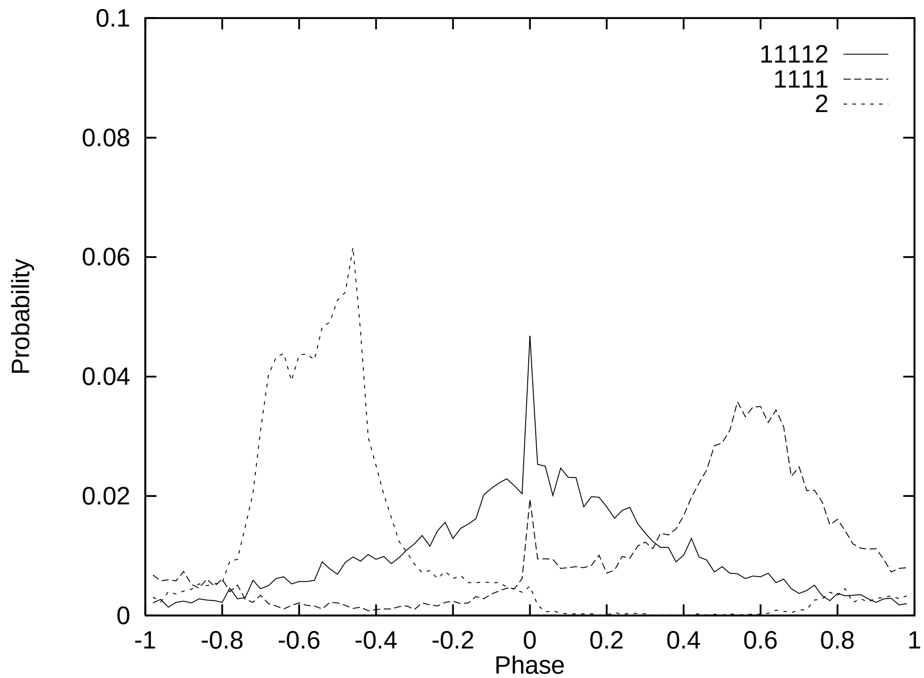

Figure 7.2 Normalized phase distribution for a gauge field with no electric field ($\phi = 0$) and $h_1 = 0.23$, $h_2 = 0.37$ on a 6×6 lattice.

evidence we see nothing qualitative left to check regarding the applicability of the overlap formalism to the zero topology sector of this new 11112 model. Recalling our study of the Schwinger model [9] we consider it very likely that no difficulties will be uncovered at non-zero topology.

We conclude that the new 11112 model very likely has the following two main properties:

- It is an exactly soluble chiral model with $U(1)$ gauge interactions and fermion number violation.
- The exact continuum results are reproduced by the overlap formalism.

To prove the above one would have to carry out a complete numerical simulation. This simulation would be similar to the ones in [9]. The expectation values of the appropriate 't Hooft vertex ($\prod_{\alpha,\beta=1}^{Q}[\bar{\psi}_{R,\alpha\beta}(x)][(\psi_L \partial_L \psi_L)(x)]$) obtained on a sequence of increasingly finer lattices representing a fixed physical size would have to be extrapolated to the continuum limit. The simulation would employ gauge averaging, but this would have almost no numerical effect since the real part of the fermion induced gauge action would be discarded. In order to know whether the result is correct, one would need the exact continuum expressions for the 't Hooft vertices in the appropriate finite Euclidean volume. The needed formulae would have to be obtained following [2].

8. Conclusions

Surprisingly, chiral abelian two dimensional gauge theories on a torus do not provide a completely transparent testing ground for the overlap lattice formalism. More specifically, the ingredient of this formalism one would like to test outside perturbation theory is the Wigner–Brillouin phase choice. All other non-perturbative ingredients can and have been tested in vectorial theories with global chiral symmetries. These tests were all successful. In the present work we focused on two versions of the 11112 chiral $U(1)$ model. What we found supports the validity of the basic strategy of [3]. This strategy employs the Wigner–Brillouin phase choice and restores exact gauge invariance by averaging along gauge orbits. It is logically based on the work of Förster–Nielsen–Ninomiya [4].

Let us first list our findings without our interpretation.

A) 11112 with antiperiodic boundary conditions for all fermions.

- There exists a large class of gauge orbits, not of zero measure, on which the gauge averaging of W-B phases appears to wash out the expected gauge invariant, parity odd, continuum answer one would obtain in a smooth background.
- This class can be identified in the continuum, independently of any knowledge about the intended regularization procedure.
- On each of the above “bad” orbits the source of the wash-out was identified. We found that the “bad” gauge transformations on the “bad” orbits approximate continuum A_μ's that have gauge singularities in the form of coherently ordered linear delta functions.
- There also exists a large class of “good” gauge orbits, again not of zero measure, on which the gauge averaging of W-B phases produces the expected gauge invariant, parity odd, continuum answer. The gauge averaging on the “good” orbits is unrestricted, including gauge transformations that would be deemed “bad” on “bad” orbits.
- On “bad” orbits, if one eliminates the “bad” gauge transformations, gauge averaging does produce the expected answer. Anomaly cancelation is essential.
- If one adds a sufficiently strong soft gauge breaking term of the type $\int(\partial_\mu A_\mu)^2$ and then uses gauge averaging along orbits all indications are that the continuum results will be fully reproduced by the overlap.

B) 11112 with special fermionic boundary conditions.

- All difficulties in the continuum and on the lattice disappear. Gauge averaging works fine.

Our interpretation, based on a fiber bundle view of the domain of integration in the path integral, is that the “bad” gauge transformations spoiling gauge invariance on the

"bad" orbits are a continuum difficulty. This view might be contested if one insists that all the allowed vector potentials in the zero topology sector are essentially smooth functions in the continuum.

Our message for the future is two-fold: The first part is independent of our interpretation. One should carry out a complete dynamical simulation of model *B)* and check whether the continuum results are reproduced quantitatively. The new fermion boundary conditions remind us of the twisted many flavor Schwinger models devised by Shifman and Smilga [10].

The second part of our message is that attention should be paid to gauge singularities that ordinary fiber bundle representations admit. A thorough investigation of chiral fermions in such backgrounds should be possible by continuum methods and might reveal differences between chiral and non–chiral anomaly free theories.

One may say that the evidence in favor of the overlap formalism accumulated to date is enough to turn the tables around and accept this formalism as a basis for the investigation of the existence of asymptotically free chiral gauge theories outside perturbation theory. If the gauge breaking terms turn out to be too strong in some particular case one can always use patches like at the end of section 6 to weaken them. If this does not work one should suspect not only the regularization but entertain the possibility that the fault is in the continuum.

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